Condition for a quadratic equation to have 2 real roots.

2 real roots

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$$ax^2 + bx + c = 0.$$

roots: $$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

2 real roots if $b^2 - 4ac > 0$.

e.g. $$x^2 - 3x + 2 = 0$$

$$x = \frac{3 \pm \sqrt{3^2 - 4(2)}}{2}$$

$$= \frac{3 \pm \sqrt{1}}{2}$$

$$= \frac{3+1}{2} \text{ or } \frac{3-1}{2}$$

$$= 2 \text{ or } 1.$$

$b^2 - 4ac$ called discriminant.

If +ve, $\sqrt{b^2 - 4ac}$ is real number.

If -ve, eg. like $\sqrt{-1}$ is imaginary number.

Then no real root.

Condition for a quadratic equation to have 2 equal roots.

2 equal roots

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$$ax^2 + bx + c = 0.$$

roots: $\quad x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

2 equal roots if discriminant $\quad b^2 - 4ac = 0.$

e.g. $\quad x^2 - 6x + 9 = 0$

$$x = \frac{6 \pm \sqrt{6^2 - 4(9)}}{2}$$

$$= \frac{6 + \sqrt{0}}{2} \quad \text{or} \quad \frac{6 - \sqrt{0}}{2}$$

$$= 3 \quad \text{or} \quad 3$$

So 2 equal roots.

If $\quad b^2 - 4ac = 0, \quad$ then $\quad \sqrt{b^2 - 4ac} = 0$

So $\quad \pm\sqrt{b^2 - 4ac} \quad$ is just $\quad \pm 0.$

No difference in answers.

Condition for a quadratic equation to have no real roots.

No real roots

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$$ax^2 + bx + c = 0.$$

roots: $$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

No real roots if discriminant $b^2 - 4ac < 0$.

e.g. $$x^2 + 2x + 3 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)}}{2}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

$\sqrt{-8}$ not real number. So no real roots.

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is square root of $-$ve number, So not real number.

condition for a given line to intersect a given curve.

Line Intersect Curve

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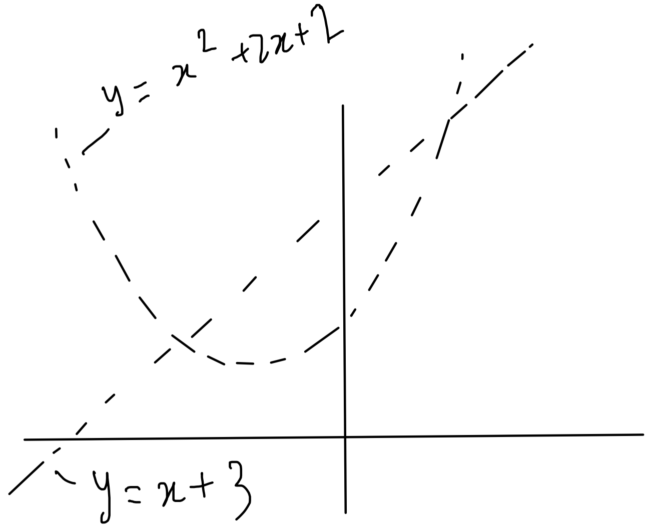
e.g. find the points of intersection of →



Method 1: plot graphs.

Method 2: solve these:

$$y = x^2 + 2x + 2$$
$$y = x + 3$$

Substituting, $x + 3 = x^2 + 2x + 2$

$$0 = x^2 + x - 1 \quad \text{(quadratic)}$$

Roots $x = \dfrac{-1 \pm \sqrt{1^2 - 4(-1)}}{2}$

$$= \frac{-1 \pm \sqrt{5}}{2} \quad b^2 - 4ac > 0$$

(2 real roots)

2 points of intersection ← 2 answers

<u>Summarise:</u>

Q. How to tell if a quadratic curve intersects a straight line?

A. When we solve the simultaneous equations, the resulting quadratic equation must have discriminant > 0.
$(b^2 - 4ac)$

condition for a given line to be a tangent to a given curve.

Line Touches Curve

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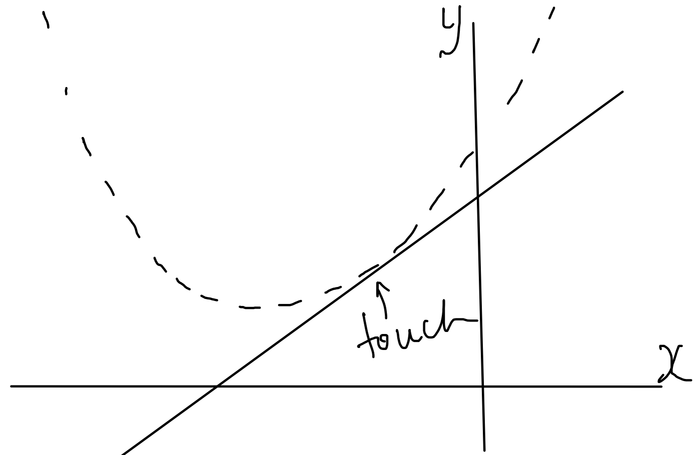
e.g. $y = x^2 + 3x + 4$
$y = x + 3$



1. Plotting graphs show they just touch.

2. Try solving: $x^2 + 3x + 4 = x + 3$

$$x^2 + 2x + 1 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)}}{2}$$

$$= \frac{-2 \pm \sqrt{0}}{2}$$ $b^2 - 4ac = 0$

2 points of intersection are the same ← 2 equal roots

↓

touch at one point!

Summarise:

Q. How to tell if a quadratic curve touches a straight line?

A. When we solve the simultaneous equations, the resulting quadratic equation must have discriminant = 0. ($b^2 - 4ac$)

condition for a given line to not intersect a given curve.

Line Not Intersect Curve

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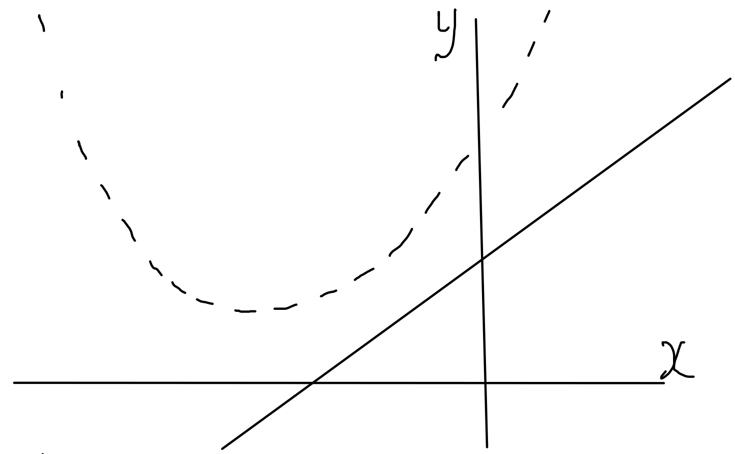
e.g. $y = x^2 + 3x + 4$
$\quad\;\; y = x + 1$



1. Plotting graphs show they do not intersect.

2. Try solving: $x^2 + 3x + 4 = x + 1$

$$x^2 + 2x + 3 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)}}{2}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

$b^2 - 4ac < 0$

↓

no point of intersection ← no real roots

↓

do not intersect!

Summarise:

Q. How to tell if a quadratic curve does not intersect a straight line?

A. When we solve the simultaneous equations, the resulting quadratic equation must have discriminant < 0.
$(b^2 - 4ac)$

Condition for ax^2+bx+c to be always positive.

Quadratic Always Positive

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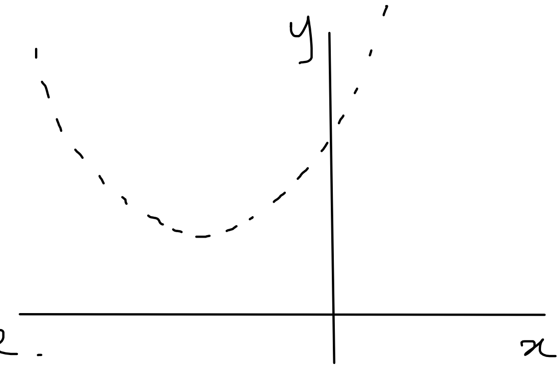
e.g $y = x^2 + 2x + 3$



Plot graph shows

x^2+2x+3 always positive.

Can we tell without plotting?

Yes - x^2+2x+3 always positive

$\Rightarrow$ $x^2+2x+3=0$ not possible, or no answer.

$\Downarrow$

no real roots!

$\therefore$ discriminant $b^2-4ac<0$.

Check: $2^2-4(3) = -8 < 0$ ✓

Summarise:

ax^2+bx+c is always positive if

$b^2-4ac<0$.

Condition for ax^2+bx+c to be always negative.

Quadratic Always Negative

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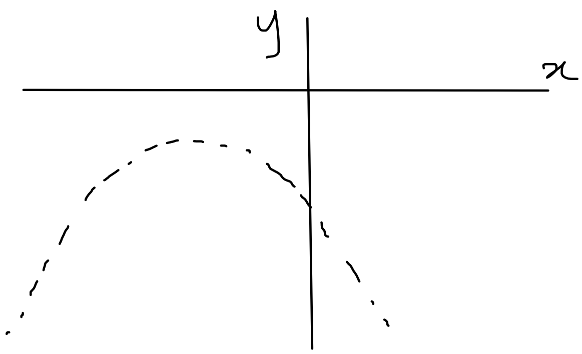
e.g $y = -x^2 - 2x - 3$



Plot graph shows

$-x^2 - 2x - 3$ always negative.

Can we tell without plotting ?

Yes - $-x^2 - 2x - 3$ always negative

$\Rightarrow -x^2 - 2x - 3 = 0$ not possible, or no answer.

$\Downarrow$

no real roots !

$\therefore$ discriminant $b^2 - 4ac < 0$.

Check : $2^2 - 4(-1)(-3) = -8 < 0$ ✓

Summarise :

$ax^2 + bx + c$ is always negative if

$b^2 - 4ac < 0$.

Solve Quadratic & Linear Equations

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e.g.

$$y = x^2 + 2x + 2 \quad \text{— (1)}$$
$$y = x + 3 \quad \text{— (2)}$$

Solution:

Substituting (1) into (2) :

$$x + 3 = x^2 + 2x + 2$$
$$0 = x^2 + x - 1 \quad \text{(quadratic)}$$

Solving:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-1)}}{2}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$

Roots & Coefficients

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e.g. $ax^2 + bx + c = 0$ — (1)

Let the roots be α, β.

Want to find equations that relate α, β to a, b, c.

Method:

1. If factorise, must get $(x-\alpha)(x-\beta) = 0$
   $\therefore$ α, β are roots.

2. Expanding: $x^2 - \beta x - \alpha x + \alpha\beta = 0$
   $x^2 - (\beta + \alpha)x + \alpha\beta = 0$ ---(2)

3. Should be same as (1). But x^2 in (1) has coefficient a.

4. Multiply (2) by a: $ax^2 - a(\alpha+\beta)x + a\alpha\beta = 0$ ---(3)

5. Now x^2 term same as (1). x and constant terms must be same also.

6. Compare coefficients of (3) and (1):

x^2: $-a(\alpha+\beta) = b$ $\Rightarrow$

constant: $a\alpha\beta = c$ $\Rightarrow$

$$\boxed{\begin{aligned} \alpha+\beta &= -\frac{b}{a} \\ \alpha\beta &= \frac{c}{a} \end{aligned}}$$

Solving quadratic inequalities, and representing the solution on the number line

Quadratic Inequality

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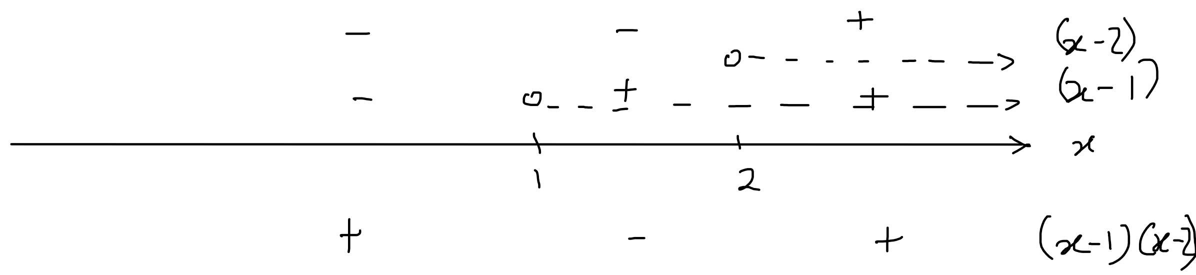
e.g. Solve $x^2 - 3x + 2 < 0$.

Ans. Factorise $(x-1)(x-2) < 0$

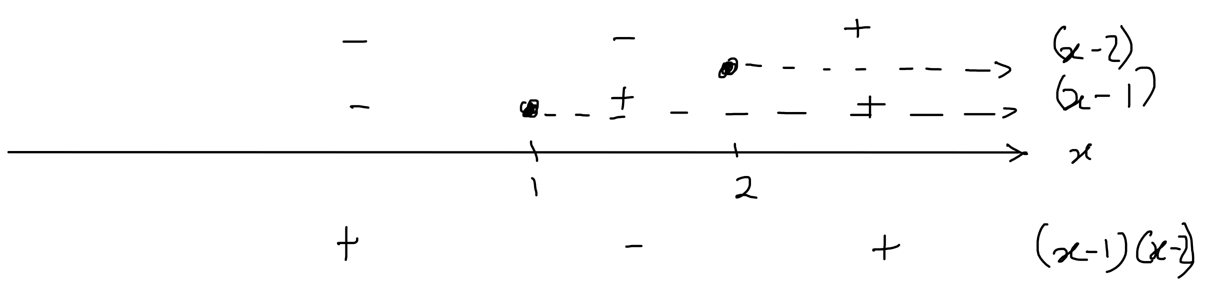


$\therefore \quad 1 < x < 2$

(not $1 < x$ or $x < 2$)

e.g. Solve $x^2 - 3x + 2 \geq 0$.

Ans. Factorise $(x-1)(x-2) \geq 0$



$\therefore \quad x \leq 1$ or $x \geq 2$

(not and)

(not $2 \leq x \leq 1$!)

Problem

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2013 P1 Q7 The roots of the equation $2x^2 + px - 1 = 0$, where p is a positive constant, are α and β. The roots of the equation $x^2 - 5x + q = 0$, where q is a constant, are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$. Find p and q.

Solution $2x^2 + px - 1 = 0 \rightarrow$

$$\alpha + \beta = -\frac{p}{2} \quad -(1)$$

$$\alpha\beta = -\frac{1}{2} \quad -(2)$$

$x^2 - 5x + q = 0 \rightarrow$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{5}{1} \quad -(3)$$

$$\frac{1}{\alpha^2\beta^2} = \frac{q}{1} \quad -(4)$$

Substitute (2) into (4): $\frac{1}{(-\frac{1}{2})^2} = q \rightarrow q = 4. \quad -(5)$

(3) $\rightarrow$ $\frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = 5 \quad -(6)$

Using (4) & (5) $\rightarrow$

$$4(\alpha^2 + \beta^2) = 5$$

$$4[(\alpha+\beta)^2 - 2\alpha\beta] = 5$$

(1), (2) $\rightarrow$

$$4[(-\tfrac{p}{2})^2 - 2(-\tfrac{1}{2})] = 5$$

$$p^2 + 4 = 5$$

$$p = \pm 1.$$

But question says p positive, so $p = 1$ only.